

Delineating the conformal window

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We identify and characterise the conformal window in gauge theories relevant for beyond the standard model building, e.g. Technicolour, using the criteria of metric confinement and causal analytic couplings, which are known to be consistent with the phase diagram of supersymmetric QCD from Seiberg duality. Using these criteria we find perturbation theory to be consistent throughout the predicted conformal window for several of these gauge theories and we discuss recent lattice results in the light of our findings.

THE CONFORMAL WINDOW

In a generic non-Abelian gauge theory with gauge group G and N_f fermions transforming according to a representation R of G we expect there to be a conformal window [1], i.e. a region $N_f^{\text{II}} < N_f < N_f^{\text{I}}$ for which the theory is asymptotically free at short distances while the long distance physics is scale-invariant and typically governed by a non-trivial fixed-point. In this paper we consider such theories with fermions in a single representation of the gauge groups SU, SO, Sp .

The upper boundary of the conformal window is determined in perturbation theory from the β function:

$$\beta(x) \equiv \frac{dx}{d\ln(Q^2)} = -(\beta_0 x^2 + \beta_1 x^3 + \dots), \quad (1)$$

at a small value of the coupling $x \equiv \alpha_s/\pi$. The first two coefficients of the expansion [2, 3] are universal and independent of the renormalisation group scheme:

$$4\beta_0 = \frac{11}{3}C_2(G) - \frac{4}{3}T(R)N_f \quad (2)$$

$$16\beta_1 = \frac{34}{3}C_2^2(G) - \frac{20}{3}C_2(G)T(R)N_f - 4C_2(R)T(R)N_f. \quad (3)$$

When β_0 changes sign, from positive to negative at

$$N_f^{\text{I}} = \frac{11}{4} \frac{C_2(G)}{T(R)}, \quad (4)$$

the theory changes from the asymptotically free conformal phase to the infrared free phase. This is the upper boundary of the conformal window, coinciding with the loss of asymptotic freedom (LOAF), and the transition point in $N_c = 3$ QCD is at $N_f^{\text{I}} = 16.5$. For N_f just below this upper boundary, Eqs.(3,4) imply that $\beta_1 < 0$, and so $\beta(x)$ will have a non-trivial zero at $x_{\text{FP}} \simeq -\beta_0/\beta_1 > 0$. The fixed point coupling x_{FP} approaches zero as N_f approaches N_f^{I} from below. The smallness of x_{FP} just below N_f^{I} justifies the use of the 2-loop β function. Thus the transition to the infrared free phase is always via a conformal phase [1] and this is independent of the fermion representation.

The lower boundary of the conformal window, N_f^{II} , below which confinement and chiral symmetry breaking

typically set in, is much harder to determine. From the two-loop β -function, the fixed point is lost and the lower boundary of the conformal window would be reached from above when $\beta_1 = 0$. However, this not only ignores higher order corrections but also neglects non-perturbative effects which, generally, are expected to become important towards the lower end of the conformal window, where the 2-loop estimate of the fixed point coupling is becoming large, $x_{\text{FP}} \gtrsim 1$.

While the lower boundary of the conformal window is of theoretical interest in its own right, its current importance arises from its central role in technicolour models [4] with walking dynamics [5, 6] and, in particular, of more recent models such as minimal walking technicolour [7] and conformal technicolour [8]. Therefore, a lot of effort has recently gone into exploring this region, using both lattice [9–24] and approximate analytical [25–37] methods. In principle the former should provide a definitive answer: however, it has become clear, from the pioneering lattice calculations, that identifying and characterising (near-)conformal theories on a lattice is a very challenging problem. So it remains important to try and gain as much analytical insight as possible.

Since it is the chiral symmetry breaking of technicolour that drives the interesting ‘walking’ scenarios, it is natural to look to analytic methods that estimate its onset. The standard technique involves the use of Schwinger-Dyson (SD) equations in a ladder-like approximation [25, 28, 29]. While this does make a prediction for the value of N_f at which chiral symmetry is spontaneously broken, the credibility of the estimate is called into question by the fact that in the case of $\mathcal{N} = 1$ supersymmetric QCD (SQCD), where Seiberg duality [38] allows us to calculate the value of N_f^{II} exactly, the SD estimate is far above the known value [39]. Thus it is useful to look for other analytical estimates which can help determine where conformality may be lost.

Here we wish to discuss two such methods, both of which have been extensively discussed in the 1990’s in related and overlapping contexts. First we shall discuss the criterion of ‘metric confinement’ [40], which provides a *lower bound* on the value of N_f at which confinement occurs and thus also for the value of N_f^{II} at which conformality is lost. Secondly we discuss the range of validity of

perturbation theory within the conformal window following [26, 41, 42] and we compare our findings with lattice simulations of these theories.

Metric confinement

Metric confinement determines when transverse gluons are not part of the physical Hilbert space from the properties of the transverse gluon propagator, $D(Q^2, \mu^2, g)$, where μ^2 is the renormalisation scale. We refer the reader to [40] for a detailed exposition of metric confinement. The condition can be formulated (working always in Landau gauge) in terms of a superconvergence relation for the absorptive part $\rho(k^2, \mu^2, g) = (1/\pi) \text{Im} \{D(-k^2, \mu^2, g)\}$ of the gluon propagator [40]:

$$\int_0^\infty dk^2 \rho(k^2, \mu^2, g) = 0. \quad (5)$$

Because of the known analyticity properties of the propagator D , Eq. (5) is equivalent to the vanishing of the integral of D around the contour at $|k^2| = \infty$ [40]. Thus, if $D(Q^2, \mu^2, g)$ vanishes fast enough as $|Q^2| \rightarrow \infty$, one will indeed have metric confinement. Asymptotic freedom then allows us to determine whether it does so or not from the value of the appropriate anomalous dimension. The condition for metric confinement, in terms of the 1-loop anomalous dimension of the gluon propagator γ_{00} can be seen to be [40]:

$$\gamma_{00} = -\frac{1}{4} \left(\frac{13}{6} C_2(G) - \frac{4}{3} T(R) N_f \right) < 0. \quad (6)$$

Note that because we are interested in the value of D as $|Q^2| \rightarrow \infty$, the 1-loop perturbative value of γ_{00} is exact for our purposes: when Eq. (6) holds the theory confines and conformality has been lost. Metric confinement is claimed to provide a sufficient but not necessary condition for confinement and therefore Eq. (6) provides a lower bound on the lower boundary of the conformal window:

$$N_f^{\text{II}} \geq N_f^{\text{MC}} \equiv 13C_2(G)/8T(R). \quad (7)$$

We also note from Eq. (4) that this bound is strictly less than the upper edge of the conformal window: $N_f^{\text{MC}} < N_f^{\text{I}}$. So metric confinement always leaves a finite window of opportunity for conformality.

This lower bound on N_f^{II} [40] is plotted for SU and SO gauge theories with fermions in single- and two-index representations, as the thick dotted line, in Figs. 1 and 2. We discuss the implications later in the paper.

Just as with the SD estimates, it is useful to test this bound in SQCD. Remarkably, one finds that the lower bound on N_f^{II} from metric confinement *coincides* with the value of N_f^{II} that is determined from Seiberg duality [38].

This has been shown for both SU and SO gauge groups [26, 43, 44] and is also the case for Sp gauge groups, as we have checked ourselves. Such agreement is particularly significant in the case of SQCD as it is known [38] that here the loss of conformality is through the onset of confinement and not of chiral symmetry breaking – the latter occurring at a much smaller value of N_f . (This provides a striking counterexample to the earlier wisdom that confinement necessarily entails chiral symmetry breaking.)

It is also interesting to consider supersymmetric Yang Mills with fermionic matter in higher representations where there is no known Seiberg dual. In these cases if one determines the lower boundary of the conformal window using the Novikov-Shifman-Vainshtein-Zakharov (NSVZ) beta function for supersymmetric theories [45] by setting $\gamma = 1$ (the unitary bound in these theories) [29], which in the case of SQCD is known to reproduce the result from Seiberg duality, we find that even in these theories metric confinement coincides with this result.

Motivated by these examples, we shall assume in the remainder of this paper that metric confinement is (usually) not just a sufficient but also a necessary condition for confinement to occur.

Perturbation theory and analyticity

At large momentum transfer Q^2 , the coupling constant behaves as $x(Q^2) \sim \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)}$. At 1-loop this simple expression is valid for all Q^2 , so that $x(Q^2)$ diverges at $Q^2 = \Lambda^2$. Thus if we attempt to calculate some physical quantity in a convergent power series in the 1-loop running coupling, this physical quantity will inherit this Landau singularity. This, however, will in general violate the known analyticity properties of such a physical quantity, which typically involves specific poles and cuts corresponding to asymptotic states. Thus we see that perturbation theory in the 1-loop running coupling cannot be adequate and that this is immediately visible from the unphysical analytic structure of the coupling. This suggests that, more generally, the analytic structure of a running coupling can indicate whether there is any possibility of perturbation theory providing a complete description of the physics.

Here we are interested in studying the conformal window and, in this case, we have an infra-red fixed point, so the coupling is bounded by $0 \leq x(Q^2) \leq x_{FP}$ for $0 \leq Q^2 < \infty$ and so cannot have such a divergence. In particular this is the case if we use the 2-loop coupling and if $\beta_1 < 0$. As we approach the upper bound, $N_f \rightarrow N_f^{\text{I}}$, the coupling becomes weak on all scales and we may expect perturbation theory to work well. In that case, the coupling $x(Q^2)$ should manifest the analytic structure of a typical physical quantity i.e. a cut for $k^2 = -Q^2 \geq 0$ corresponding to the production of mass-

less particles, and no other unphysical singularities in the entire complex Q^2 plane. If this is so then it is said to be *causal analytic* and indeed this turns out to be the case for $N_f \rightarrow N_f^I$ [41]. If we now decrease N_f away from N_f^I then, as long as the coupling remains causal analytic, it is consistent for the physics to be perturbative. As we continue decreasing N_f , at some point $x(Q^2)$ will acquire unphysical singularities in the complex Q^2 plane. These might be poles or cuts. At this point the coupling ceases to be causal analytic and signals the fact that there must now be non-perturbative contributions that will serve to restore the correct analytic structure to the quantity being calculated. These may lead to confinement and/or chiral symmetry breaking and hence the loss of conformality.

The two loop β -function can be integrated explicitly in terms of the Lambert W-function [46] defined by $W(z) \exp[W(z)] = z$, giving [26, 41, 42]

$$x(Q^2) = -\frac{1}{c} \frac{1}{1 + W(z)}, \quad c = \frac{\beta_1}{\beta_0},$$

$$z = -\frac{1}{ce} \left(\frac{Q^2}{\Lambda^2} \right)^{-\beta_0/c}.$$

While $W(z)$ is a multi-valued function with an infinite number of branches, the unique branch for $c < 0$ with a real coupling along the positive real Q^2 axis is the principal branch denoted $W_0(z)$ [26, 41, 42]. The requirement for this coupling to be causal translates into the criterion

$$0 < -\beta_0^2/\beta_1 < 1 \quad (8)$$

Note that as one approaches the upper bound to the conformal window, $\beta_0 \rightarrow 0^+$ while $\beta_1 < 0$, this bound is always satisfied, i.e. the coupling is causal analytic in this Bank-Zaks limit, as one might expect. Note also that this is a stronger criterion than just requiring that the two-loop β -function have a fixed-point since, as $\beta_1 \rightarrow 0^-$ one violates the bound in Eq. 8. Reflecting this, the analytically continued coupling will acquire singularities in the complex plane at a larger value of N_f than where the Landau singularity appears [26, 41, 42].

We observe from Eq. 8 that the coupling is causal analytic all the way down to N_f^{MC} provided $C_2(R) > \frac{11}{26}C_2(G)$, which is true in all cases, except for $SU(2)$ (and $Sp(4)$) with fundamental fermions. Hence it is also the case all the way down to N_f^{II} if we accept the bound in Eq. (7). For multi-flavor QCD this was already noted in [26]. This demonstrates that while causal analyticity may be a necessary condition for non-perturbative physics to be unimportant, it is not sufficient. In [26] it was also shown that in SQCD (whose β -function differs from Eqs. (2,3) because of the presence of scalars and gluinos) analyticity breaks down *before* N_f^{II} is reached. This fits in with the requirements of the weak-strong coupling Seiberg duality [38] where the lower and upper

boundaries of the conformal windows of the dual theories are mapped into each other, which implies that near the lower boundary the theory must be strongly coupled. This demonstrates that when analyticity breaks down, so that non-perturbative physics must be present, this does not necessarily entail confinement, chiral symmetry breaking, or indeed the loss of conformality.

The analyticity bound in Eq. (8) is obtained from the 2-loop β -function and so can only be regarded as approximate. (Although in [42] it was shown that going to 3-loops, utilising a particular Padé approximant functional form, does not alter the conclusions, as long as the 3-loop coefficient of the β -function is not very large.) Moreover, we expect that the perturbative expansion for $\beta(x)$ cannot be better than asymptotic, with corrections $\sim \exp\{-c/x\}$ that mimic non-perturbative contributions. Roughly speaking, we would expect the causal analyticity calculated at 2-loops to be reliable as long as the coupling $x(Q^2)$ is not too large anywhere in the complex Q^2 plane.

When judging whether a coupling is 'small' or 'large' it is in some sense more natural to use the scaled ('t Hooft) coupling $N_c x$ instead of x as, at large N_c , $x \sim N_c^{-1}$ while the n -th coefficient of the β -function scales as $\beta_n \sim N_c^{n+1}$, and similarly for the anomalous dimension. As an example, the mass anomalous dimension of an adjoint fermions is given by $\gamma_{\text{Adj}} = \frac{3}{2}(N_c x) + O(N_c^2 x^2)$. We shall therefore calculate $\max_{Q^2 \in \mathbb{C}} |N_c x(Q^2)|$ using the correct analytic continuation of x from the 2-loop β -function and use the magnitude of the result as a supplementary criterion for judging the reliability of any argument from analyticity.

For the moment we simply plot the value of N_f where analyticity is lost, and hence where perturbation theory signals its own breakdown according to the criterion in Eq. (8), as the black solid lines in Figs. 1 and 2. We interpret these results below.

Analyticity with the all-orders beta-function conjecture

Inspired by the NSVZ beta function [45], an all-orders (AO) beta function for $SU(N)$ gauge theories with any matter representation was conjectured in [30] and further studied in [34]. It reads:

$$\beta(x) = -\beta_0 x^2 \frac{1 - T(R) N_f \gamma(x)/(6\beta_0)}{1 - \frac{x}{2} C_2(G) \left(1 + \frac{2\beta'_0}{\beta_0}\right) x}, \quad (9)$$

where,

$$\gamma(x) = \frac{3}{2} C_2(R) x + O(x^2), \quad 4\beta'_0 = C_2(G) - T(R) N_f \quad (10)$$

Here, $\gamma \equiv -\frac{d \ln m}{d \ln \mu}$ is the fermion mass anomalous dimension, and solving for γ at a fixed point, i.e $\beta = 0$, yields $\gamma = \frac{11C_2(G) - 4T(R)N_f}{2T(R)N_f}$ which increases as N_f is decreased.

Since $\gamma \leq 2$ is a rigorous bound from unitarity [47], this provides a different *lower bound* on N_f^{II} ,

$$N_f^{\text{AO}} = \frac{11}{8} \frac{C_2(G)}{T(R)} \quad (11)$$

which we see is slightly below the bound provided by metric confinement in Eq. (7).

In Figs. 1 and 2 we plot this lower bound, N_f^{AO} , as a thick dashed line. For the adjoint representations this line is invisible because it exactly coincides with the thick solid line that represents the loss of causality in the two-loop β -function.

We observe that if we restrict the matter anomalous dimension γ to first order in x then this all orders β -function may be integrated exactly, yielding:

$$x(Q^2) = \frac{1}{E_1} \frac{1}{1 + G_1 W(z)}, \quad G_1 \equiv 1 - \frac{D}{E_1},$$

$$z = \frac{1}{G_1} \exp(-1/G_1) \left(\frac{Q^2}{\Lambda^2} \right)^{\frac{\beta_0}{E_1 G_1}},$$

where

$$E_1 = C_2(r)T(R)N_f/(4\beta_0), \quad D = \frac{1}{2}C_2(G) \left(1 + \frac{2\beta'_0}{\beta_0} \right).$$

We can integrate the AO β -function in this approximation of γ as it has the same structure as a Padé approximant to the 3-loop β -function which is integrable in terms of the W -function [42]. The condition for having a causal coupling thus becomes $\beta_0 < E_1 - D$ which is identical to the criterion for the two loop coupling being causal.

Similarly the coupling is causal analytic all the way down to N_f^{AO} provided $C_2(R) > \frac{199}{198}C_2(G)$, which for the theories considered here, is generally only the case for the two-index symmetric representation.

Comparing with lattice data and other methods

Both the criterion of metric confinement and that of causal analyticity are consistent with the properties of the conformal window in SQCD as predicted from Seiberg duality. It is therefore interesting to ask what these criteria predict for the non-supersymmetric theories that are being investigated using lattice techniques. These theories include $SU(2)$ and $SU(3)$ with a ‘large’ number of fundamental (F) fermions [14–19], $SU(2)$ with 2 adjoint (Adj) fermions [9–13], and $SU(3)$ with 2 sextet (2S) fermions [20–24]. These theories are part of the larger family of theories whose properties are shown in Figs. 1 and 2. On each of these plots we show N_f^{I} , as well as three curves related to the lower boundary of the conformal window: the curve N_f^{MC} where metric confinement sets in, the curve N_f^{AO} mapped out by the vanishing of the AO β -function with $\gamma = 2$, and the curve

where causal analyticity breaks down. The first two provide lower bounds for the conformal window, while the third gives us an estimate of where non-perturbative effects must be important. We have also displayed in these figures the SD predictions for chiral symmetry breaking (in the usual ladder approximation). Where chiral symmetry breaking occurs will typically be the lower boundary of the conformal window and, in any case, will provide a lower bound for it. Unfortunately, although time-honoured, such SD estimates are known to fail in SQCD [39].

$SU(2)$ and $SU(3)$ theories with fundamental flavours

In the left panels of Figs. 1 and 2 we display estimates for the conformal window of SU and SO theories (Sp being qualitatively the same as SU) with fundamental fermions. It shows that the metric confinement and causal analyticity criteria almost coincide in all cases. With the exception of $SU(2)$ (and $Sp(4)$), causal analyticity extends to a slightly lower N_f than metric confinement. So, in contrast to SQCD, the whole of the conformal window is causal analytic, suggesting that it represents a perturbative infra-red conformal phase.

For $SU(3)$ this suggests that the conformal window begins with $N_f = 10$ and for $SU(2)$ with $N_f = 7$. However, since the limits are close together it is important to check whether the coupling remains small at these limits. In Fig. 3 we plot the maximal value of the complex 2-loop coupling $\max_{Q^2 \in \mathbb{C}} |N_c x(Q^2)|$ for $SU(3)$, as a function of the scaled flavour variable $\Delta N_f \equiv (N_f - N_f^{\text{MC}})/(N_f^{\text{I}} - N_f^{\text{MC}})$ taking values from 0 to 1 within the conformal window, and indicate with dots the $N_f = 10, 12, 16$ theories. We see that, as expected the coupling remains small for $N_f = 16$ and increases as N_f is lowered. In particular, the coupling is rather large at the lower end of the window, leaving room for a significant shift, either way, in our estimate of what is the true region of causal analyticity.

Inside the conformal window the coupling does not decrease linearly with N_f but rather increases rapidly as N_f^{MC} is approached. This behaviour is plotted in Fig. 3. Although in $SU(3)$ the coupling rapidly increases below $N_f = 10$ it should be noted that the coupling is already somewhat large by this point.

The so-called 1-family models of technicolour are based on an $SU(2)$ gauge theory with $N_f = 8$ in the fundamental representation, see e.g [48]. This theory is well above the bound on N_f^{MC} that follows from metric confinement and within the window of causal analyticity with a relatively small coupling shown in Fig. 3, suggesting that the theory is conformal and weakly coupled.

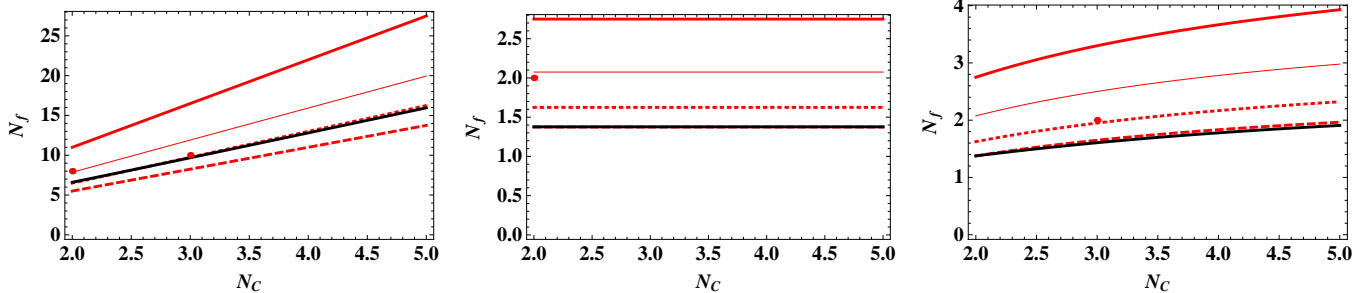


FIG. 1: Conformal windows for SU theories with Dirac fermions in the fundamental (left), adjoint (mid) and two-index symmetric (right) representations. On all three figures the curves indicate N_f^I (thick upper solid) and N_f^{II} according to SD (thin solid), metric (thick dotted), AO β -function with $\gamma = 2$ (thick dashed) and finally loss of causal analyticity (thick lower solid, black). For the adjoint representation the latter two very nearly coincide. The theories discussed in the main text are indicated with red dots.

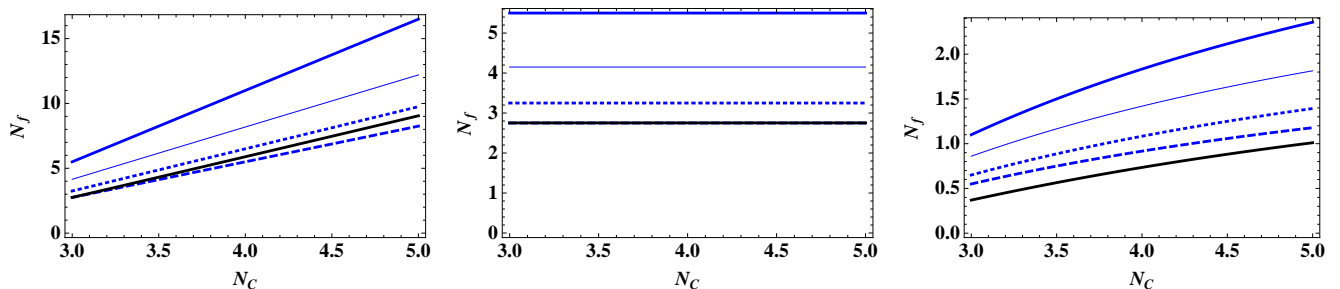


FIG. 2: Same as Fig. 1 but for SO theories.

Two flavor $SU(2)$ adjoint theories

The Minimal Walking technicolour (MWT) model [7, 49] is based on $SU(2)$ gauge theory with $N_f = 2$ in the adjoint representation. Current lattice simulations of this theory suggests that it is conformal [9–12] with a relatively small anomalous mass dimension, close to the 1-loop estimate [12].

We display in the centre panels of Figs. 1 and 2 what happens for gauge theories with adjoint fermions. We do so for various values of N_c , and for SO as well as the SU groups that lattice calculations have so far focused upon. Results for Sp are identical to those of SU . We note that the results look similar for the SU and SO groups and that there is no dependence on N_c for a fixed number of adjoint fermions. This is no surprise, since all our predictions involve some aspect of the perturbative running. Finally, and most interestingly, we see from Fig. 1 that $N_f = 2$ is well above the bound on N_f^{MC} that follows from metric confinement and also well within the window of causal analyticity. (Which here coincides with N_f^{AO} , the $\gamma = 2$ bound from the AO β -function.) This strongly suggests that the $N_f = 2$ theory is conformal.

One might be perturbed by the fact that, as we see in Fig. 1, causal analyticity extends into the region where metric confinement already holds. However, the gap between the two curves is small and is presumably consis-

tent with the uncertainty that higher order corrections would bring to the location of the breakdown of causal analyticity. Following on from the fundamental case, we calculate the value of x over the whole complex Q^2 plane, so as to see if it is everywhere ‘small’ and that our 2-loop analysis can be trusted or if it is somewhere ‘large’, increasing the uncertainty in our analysis.

The result $\max_{arg(Q^2)} |x(Q^2)|$ for the maximum value of $|x|$ at fixed $|Q^2|$ for the interesting case of $N_f = 2$ is shown in Fig. 4 and $\max_{Q^2 \in \mathbb{C}} |N_c x(Q^2)|$ for general N_f in Fig. 3. We observe that, while the maximum value of $|x(Q)|$ for $N_f = 2$ is not as small as it is near the $N_f^I = 2.75$ LOAF limit, it is certainly small compared to its value at the point near which causality is lost, $N_f^{CA} = 1.38$. This gives us confidence that at $N_f = 2$ the theory really is causally analytic and that it is in a perturbative (infra-red) conformal phase. It is thus consistent with the observation [12] that γ is close to the one-loop prediction.

On the other hand, at $N_f = 1.5$ the value of $|x(Q)|$ is large enough that it is entirely plausible that a higher order calculation could shift the loss of analyticity from just below that value of N_f to above it, so ensuring that metric confinement does not take place within the region of causal analyticity.

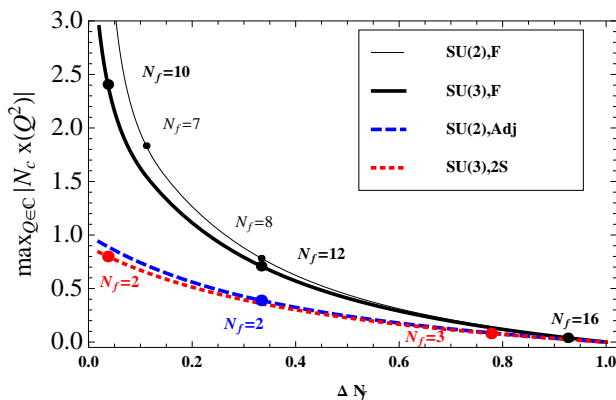


FIG. 3: The maximal value of the 2-loop coupling $|N_c x(Q)|$ in the complex plane $Q \in \mathbb{C}$, excluding the negative real axis, with $\Delta N_f \equiv (N_f - N_f^{MC})/(N_f^I - N_f^{MC})$ taking values from 0 to 1 within the conformal window for the gauge groups and representations indicated. The location of the theories of Fig. 4 are indicated in dots.

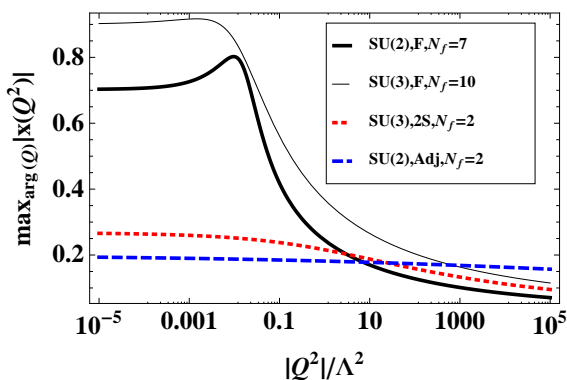


FIG. 4: The maximal value of the 2-loop coupling $|x(Q)|$ in the complex $Q \in \mathbb{C}$ plane, excluding the negative real axis, for the theories indicated. A maximum away from $|Q^2| = 0$ indicates that the theory is close to the limit of causal analyticity.

Two flavor $SU(3)$ sextet theory

The Next to Minimal Walking technicolour (NMWT) model [7, 50] is based on an $SU(3)$ gauge theory with $N_f = 2$ in the two-index symmetric (sextet) representation. Current lattice simulations of this theory suggests that it is conformal or near-conformal [21, 23, 24] and that it has relatively small anomalous mass dimension, close to the 1-loop estimate [24].

We show in the right panels of Figs. 1 and 2 what happens for SU and SO gauge theories with fermions in the two-index symmetric representation at various values of N_c (The symmetric representation of Sp is identical to the adjoint of Sp). We note that there is a significant dependence on N_c and that once again metric confine-

G	R	N_f^{CA}	N_f^{MC}	N_f^I
$SU(2)$	F	6.60	6.5	11
	Adj	1.38	1.63	2.75
$SU(3)$	F	9.68	9.75	16.5
	$2S$	1.61	1.95	3.3

TABLE I: The N_f values for loss of causal analyticity N_f^{CA} , the lower boundary of the conformal window from metric confinement N_f^{MC} , and loss of asymptotic freedom N_f^I for theories considered in the text.

ment sets in within the analyticity window. However, in contrast to the $SU(2)$ case with adjoint fermions, metric confinement sets in very close to $N_f = 2$. (See table I). Thus we expect that the $N_f = 2$ theory is very close to the lower boundary of the conformal window.

Once again we compute the value of $|x(Q)|$ from the 2-loop β -function in the whole of the Q^2 complex-plane, but this time for $SU(3)$ with 2 sextet fermions. The result for $\max_{arg(Q^2)} |x(Q^2)|$ is shown in Fig. 4 for $N_f = 2$ and $\max_{Q^2 \in \mathbb{C}} |N_c x(Q^2)|$ for general N_f in Fig. 3, where we also indicate $N_f = 3$ which is near the upper boundary of the conformal window. We observe that the maximum value of $|N_c x|$ for $N_f = 2$ is relatively small, compared to the $SU(3)$ theory with 10 fundamental flavors, although significantly larger than it is in the case of adjoint fermions. The corresponding value of $\alpha_s = \pi x$ is also larger than the value $\alpha_s \sim 0.5$ at which, in QCD, one typically begins to worry about the convergence of perturbation theory, while for MWT the coupling is indeed slightly smaller. (Though, it is not obvious how to compare the size of the couplings across theories with fermions in different representations.)

This leaves it unclear whether, at the point at which metric confinement sets in and conformality is lost, the theory is still consistently perturbative.

Conclusions

In this letter we have discussed the implications of ‘metric confinement’ and ‘causal analyticity’ for theories that are being actively studied using lattice techniques in the search for walking near-conformal field theories.

We noted that in the case of SQCD, where Seiberg duality gives us a precise description of the conformal window, both these criteria work very well: metric confinement predicts the precise location of the lower boundary of that window while causal analyticity predicts that the theory becomes strongly coupled in the lower part of the window, as required by the weak-strong duality. On the other hand, the widely used SD calculations for

where chiral symmetry breaking sets in, are very badly off in SQCD. This is part of our motivation for bringing these other criteria into play.

It is interesting that for the theories considered here, generically perturbation theory is consistent all the way down to the lower end of their conformal window as determined by metric confinement, and so the mass anomalous dimension at the fixed point can be plausibly estimated in 1-loop perturbation theory. Doing so we find $\gamma(x_{\text{FP}}) = 0.6, 1.34$ for the MWT and NMWT theories respectively. Going to the next order in \overline{MS} the values of γ change by about 10% while the corresponding predictions from the AO β -function, setting $\beta(x_{\text{FP}}) = 0$ in Eq. 9 are $\gamma(x_{\text{FP}}) = 0.75, 1.3$. This can be compared to the results of lattice simulations [12, 13, 22, 24] which suggest anomalous dimensions consistent with the 1-loop result, albeit with the caveat that for the MWT model the simulations find a fixed point which is a factor two smaller than the two-loop result we have used.

In the case of MWT both criteria suggest that this theory lies well within a perturbative infra-red conformal phase. By contrast, NMWT appears to be almost on the boundary of the lower conformal window. This is certainly consistent with the mixed messages one has been getting from different lattice calculations on this theory [22–24]. The possibility that this theory lies just outside the conformal window, which is possible because, strictly speaking, metric confinement provides a lower bound on where confinement sets in, makes it an interesting candidate walking technicolour model in itself. For example, the presence of four fermion operators, arising from extended technicolour interactions, can modify the conformal window and anomalous dimensions (indeed it can do so in all the theories we consider here[36]).

As already observed in [26], metric confinement suggests that the conformal window for $SU(3)$ with N_f fundamental fermions begins at $N_f = 10$, as we can infer from Fig. 1. As pointed out in [26] causal analyticity extends just below $N_f = 10$, suggesting that the whole conformal window is weakly coupled. However if one actually looks at the coupling x in the $N_f = 10$ theory, one finds that its value is quite large, as shown in Fig. 4. So if it turns out that the $N_f = 10$ theory does not, in fact, lie in the conformal window then again this opens the possibility of the kind of large anomalous dimension that walking phenomenology needs. On the other hand, there appears to be little doubt that the $N_f = 12$ theory does lie well inside the conformal window, and $N_f = 9$ well outside.

Very similar remarks apply to $SU(2)$ with N_f fundamental fermions. The conformal window should begin at $N_f = 7$, which is similar to $N_f = 10$ in $SU(3)$. $N_f = 8$ is very similar to $N_f = 12$ in $SU(3)$, while $N_f = 6$ lies just inside the region of metric confinement, albeit still in the region of causal analyticity.

Acknowledgements

MTF thanks E. Gardi for the initial discussion that led to this study and acknowledges a Villum Kann Rasmussen Foundation Fellowship. We thank D. D. Dietrich, F. Sannino and R. Zwicky for useful discussions.

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